

Some new almost sure results on the functional increments of the uniform empirical process

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Abstract

Given an observation of the uniform empirical process α_n , its functional increments $\alpha_n(u + a_n \cdot) - \alpha_n(u)$ can be viewed as a single random process, when u is distributed under the Lebesgue measure. We investigate the almost sure limit behaviour of the multivariate versions of these processes as $n \rightarrow \infty$ and $a_n \downarrow 0$. Under mild conditions on a_n , a convergence in distribution and functional limit laws are established. The proofs rely on a new extension of usual Poissonisation tools for the local empirical process.

Key words: Empirical processes, Functional limit theorems.

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1 Introduction and statement of the results

In 1992, Wschebor [29] discovered the following property of the Wiener process \mathbf{W} on $[0, 1]$: almost surely, for each $0 \leq a < b \leq 1$, and for each Borel set $B \subset \mathbb{R}$,

$$\lambda\left(\left\{u \in [a, b], \epsilon^{-1/2}(\mathbf{W}(u + \epsilon) - \mathbf{W}(u)) \in B\right\}\right) \xrightarrow{\epsilon \rightarrow 0} (b - a) \mathbb{P}(\mathcal{N}(0, 1) \in B). \quad (1.1)$$

Here λ denotes the Lebesgue measure. That result was later extended to a much wider class of processes by Azaïs and Wschebor [1]. It is well known that the increments of the uniform empirical process share several asymptotic

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behaviors with the increments of \mathbf{W} . Due to the works of many researchers in the past decades, our knowledge on these local functional increments (and also their generalized versions, when indexing by classes of functions, and when the distribution of the $(U_i)_{i \geq 1}$ is non necessarily uniform) is getting deeper and deeper. Strong approximation techniques of these local empirical processes by Gaussian processes (see [20], [15], [14], [6]) and Poisson processes (see [8], [11], [21]) have been established, as well as large deviation principles and functional laws of the iterated logarithm of Strassen type (see [9], [18], [13], [10], [19]) and of so called nonstandard type (see [9], [11], [28], [21]). Second order results such as clustering rates and Chung-Mogulski laws have been also established (see [5], [7], [4], [2], [3], [23]). Of course, the preceding list is non exhaustive, as the complete study of the increments of non functional type (due to the works of Mason, Csörgő, Révész, Stute among others) was also pioneering in that field. We refer to [24], Chapter 14, and the references therein for an overview on that specific topic.

A natural question is : can we obtain results such as (1.1) for these empirical increments. In this article, we provide a positive answer. In addition, we show that :

- A similar almost sure convergence in distribution holds for the **functional** increments of the uniform empirical process (see Theorem 1);
- An analogue of the Strassen law of the iterated logarithm for the functional increments of the empirical process (see Mason [18]) holds the present context (see Theorem 2);
- An analogue of the nonstandard functional law for the increments of the empirical process (see [8]) also holds (see Theorem 3);
- Each of these result hold when handling the increments of the uniform empirical process based on a **multivariate** sample.

Before stating our results, we need to introduce some notations. Denote by $D([0, 1]^d)$ the space of all distribution functions of finite signed measures on $[0, 1]^d$, and $\| \cdot \|_{[0, 1]^d}$ the sup norm on $[0, 1]^d$, namely :

$$\| f \|_{[0, 1]^d} := \sup_{t \in [0, 1]^d} | f(t) | .$$

For $f \in D([0, 1]^d)$ and $A \subset [0, 1]^d$ Borel, we shall write $f(A)$ for $\mu(A)$, where μ is the measure associated to f .

Consider an i.i.d. sequence $(U_n)_{n \geq 1}$ uniformly distributed on $[0, 1]^d$. For $s = (s^{(1)}, \dots, s^{(d)})$ and $t = (t^{(1)}, \dots, t^{(d)})$ belonging to \mathbb{R}^d the notation $s \prec t$ shall be understood as $s^{(k)} \leq t^{(k)}$ for each $k = 1, \dots, d$. We shall also write the cube $[s, t] := [s^{(1)}, t^{(1)}] \times \dots \times [s^{(d)}, t^{(d)}]$. For fixed $u \in \mathbb{R}^d$ and $a \in [0, 1]$ we will

denote by $u + a$ the vector $(u_1 + a, u_2 + a, \dots, u_d + a)$ and define:

$$\Delta_n(u, a, \cdot) := \frac{\sum_{i=1}^n \left(\mathbb{1}_{[u, u+a]}(U_i) - \lambda([0, a]) \right)}{\sqrt{na^d}}.$$

We shall also write W for the standard Wiener sheet (namely $\text{Cov}(W(t), W(s)) := (s^{(1)} \wedge t^{(1)}) \times \dots \times (s^{(d)} \wedge t^{(d)})$) and λ^* (resp. λ_*) the outer (resp. inner) Lebesgue measure on the subsets of $[0, 1]^d$. Our first result is a multivariate, functional analogue of (1.1).

Theorem 1 *Assume that :*

$$a_n \downarrow 0, \quad na_n^d \uparrow \infty, \quad \liminf_{n \rightarrow \infty} \log(1/a_n) / \log \log(n) > 1. \quad (1.2)$$

Then almost surely, for each hypercube I fulfilling both $\lambda(I) > 0$ and $I \subset [0, 1 - \delta]^d$ for some $\delta > 0$, the following assertions are true :

(i) *for each closed set $F \subset D([0, 1]^d)$ we have*

$$\limsup_{n \rightarrow \infty} \frac{\lambda^*\left(\{u \in I, \Delta_n(u, a_n, \cdot) \in F\}\right)}{\lambda(I)} \leq \mathbb{P}(W \in F),$$

(ii) *for each open set $O \subset D([0, 1]^d)$ we have*

$$\liminf_{n \rightarrow \infty} \frac{\lambda_*\left(\{u \in I, \Delta_n(u, a_n, \cdot) \in O\}\right)}{\lambda(I)} \geq \mathbb{P}(W \in O). \quad (1.3)$$

Our second result is a functional law of the iterated logarithm, in the same vein as Theorem 1. We will denote by J the rate function related to the large deviation properties of a Wiener sheet

$$J(f) := \inf \left\{ \int_{[0, 1]^d} g^2(u) du, \quad f = \int_{[0, \cdot]} g(s) ds \right\}, \quad f \in D([0, 1]^d), \quad (1.4)$$

with the convention $\inf_{\emptyset} = +\infty$. Definition (1.4) enables us to write the Strassen ball as

$$\mathcal{S} := \left\{ f \in D([0, 1]^d), \quad J(f) \leq 1 \right\}. \quad (1.5)$$

Theorem 2 *Assume that :*

$$a_n \downarrow 0, \quad na_n^d \uparrow \infty, \quad \frac{na_n^d}{\log \log(n)} \rightarrow \infty, \quad \liminf_{n \rightarrow \infty} \frac{\log(1/a_n)}{\log \log(n)} > 2. \quad (1.6)$$

Then almost surely, for each hypercube I fulfilling both $\lambda(I) > 0$ and $I \subset$

$[0, 1 - \delta]^d$ for some $\delta > 0$ we have :

$$\frac{\lambda\left(\left\{u \in I, \frac{\Delta_n(u, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \rightsquigarrow \mathcal{S}\right\}\right)}{\lambda(I)} = 1. \quad (1.7)$$

Here $f_n \rightsquigarrow \mathcal{S}$ means that the sequence $(x_n)_{n \geq 1}$ has cluster set \mathcal{S} in the Banach space $D([0, 1]^d)$.

Our third result is a nonstandard functional law of the iterated logarithm, when $na_n^d \sim c \log \log(n)$ for a constant $0 < c < \infty$. To state this, we shall introduce the following rate function ruling the large deviations of a standard Poisson process on \mathbb{R}^d .

$$\mathfrak{J}(f) := \inf \left\{ \int_{[0,1]^d} h(u) du, f = \int_{[0,\cdot]} g(s) ds \right\}, f \in D([0, 1]^d), \quad (1.8)$$

with $h(x) := x \log(x) - x + 1$ for $x > 0$ and $h(0) := 0$. For a constant $c > 0$ we shall write

$$\Gamma_c := \left\{ f \in D([0, 1]^d), \mathfrak{J}(f) \leq 1/c \right\}.$$

Theorem 3 Assume that $na_n^d \sim c \log \log(n)$ for some $0 < c < \infty$. Then almost surely, for each hypercube I fulfilling both $\lambda(I) > 0$ and $I \subset [0, 1 - \delta]^d$ for some $\delta > 0$ we have

$$\frac{\lambda\left(\left\{u \in I, \frac{\Delta F_n(u, a_n, \cdot)}{c \log \log(n)} \rightsquigarrow \Gamma_c\right\}\right)}{\lambda(I)} = 1, \quad (1.9)$$

where

$$\Delta F_n(u, a_n, t) := \sum_{i=1}^n \mathbf{1}_{[u, u+a_n t]}(U_i), \quad u, t \in [0, 1]^d. \quad (1.10)$$

In each of our proofs, we systematically use two kinds of key arguments :

- A tool for replacing probabilities involving the studied processes by probabilities involving their *poissonised* versions. These Poissonised versions have a property that play the same role as the independence of increments (which plays a crucial role in the result of Wschebor [29]).
- The existing knowledge of the asymptotic behavior of probabilities for a **single** sequence of functional increments (for example the Poissonised version of $\Delta_n(0, a_n, \cdot)$).

For the proof of Theorem 1 (see §3), we only use existing results. In particular, we make use of a "Poissonisation" tool of Giné *et. al.* (see §2). For the proofs

of Theorems 2 and 3 (see §4 and §5 respectively), we need an extended version of the just mentioned Poissonisation tool, which allows us to handle maximal inequalities for sums of i.i.d. processes (those inequalities playing a crucial role in the proofs of the functional laws of Mason [18] and Deheuvels and Mason [8]). This extended version is stated and proved in §2.

2 An extended poissonisation tool

Whenever possible, substituting empirical processes by their Poissonised versions can be very handy, due to the main property of Poisson measures, which can be seen as a generalization of independence of increments for real indexed processes. The following fact, due to Giné-Mason-Zaitsev is, to the best of our knowledge, the most general form of such a Poissonisation lemma, for which the early versions go at least to Einmahl [12].

Fact 1 (Giné-Mason-Zaitsev, [17], Lemma 2.1) *Let (D, \mathcal{D}) be a measurable semigroup, $X_0 \equiv 0$ and $(X_i)_{i \geq 1}$ be a sequence of \mathcal{D} -measurable, independent, identically distributed random variables. Let η be a Poisson variable with expectation n independent of $(X_i)_{i \geq 1}$ and let $B, C \in \mathcal{D}$ be such that $\mathbb{P}(X_1 \in B) \leq 1/2$. Then*

$$\mathbb{P}\left(\sum_{i=1}^n \mathbb{1}_B(X_i)X_i \in C\right) \leq 2 \mathbb{P}\left(\sum_{i=1}^{\eta} \mathbb{1}_B(X_i)X_i \in C\right).$$

As a consequence, for any positive measurable function H we have

$$\mathbb{E}\left(H\left(\sum_{i=1}^n \mathbb{1}_B(X_i)X_i\right)\right) \leq 2 \mathbb{E}\left(H\left(\sum_{i=1}^{\eta} \mathbb{1}_B(X_i)X_i\right)\right).$$

That fact is crucial in our proof of Theorem 1. To prove Theorem 2, we shall need an analogue of Fact 1 for probabilities related to **maximal inequalities** for partial sums in a Banach space. This analogue is indeed a consequence of a much wider generalization of Fact 1, for which we need to introduce some notations. Given a semigroup D we shall write $\widetilde{D} := \bigcup_{n \geq 1} D^n$. Also, given a set χ we call a *truncating* application any function $\phi : \widetilde{D} \mapsto \chi$ for which, for any $p \geq 2$ and $d_1, \dots, d_p \in D^p$ we have $\phi(d_1, \dots, d_p, d_p) = \phi(d_1, \dots, d_p)$ and $\phi(d_1, d_1, d_2, \dots, d_p) = \phi(d_1, d_2, \dots, d_p)$. We shall say that ϕ is *zero-irrelevant* when we add the property $\phi(0, d_1, \dots, d_p) = \phi(d_1, \dots, d_p)$. We shall write, for

simplicity of notations,

$$\begin{aligned} \sum_{i=q}^{\rightarrow p} d_i &:= (d_q, d_q + d_{q+1}, \dots, \sum_{i=q}^p d_i), \text{ when } p \geq q, \\ &:= 0 \text{ otherwise.} \end{aligned}$$

when $p \geq q$ and 0 otherwise.

Proposition 2.1 *Endow \widetilde{D} with the σ -algebra $\widetilde{\mathcal{D}} := \{\widetilde{C} \subset \widetilde{D}, \forall n \geq 1, \widetilde{C} \cap D^n \in \mathcal{D} \otimes^n\}$. Let (χ, \mathcal{A}) be a measurable space and $\phi : (\widetilde{D}, \widetilde{\mathcal{D}}) \mapsto (\chi, \mathcal{A})$ a measurable truncating application. For any $B \in \mathcal{D}$, $C \in \mathcal{A}$ such that $\mathbb{P}(X \in B) \leq 1/2$ we have*

$$\mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow n} \mathbb{1}_B(X_i)X_i\right) \in C\right) \leq 2 \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow \eta} \mathbb{1}_B(X_i)X_i\right) \in C\right).$$

As a consequence, for any positive measurable function H we have

$$\mathbb{E}\left(H\left(\phi\left(\sum_{i=1}^{\rightarrow n} \mathbb{1}_B(X_i)X_i\right)\right)\right) \leq 2 \mathbb{E}\left(H\left(\phi\left(\sum_{i=1}^{\rightarrow \eta} \mathbb{1}_B(X_i)X_i\right)\right)\right).$$

Proof of Proposition 2.1:

To prove that proposition we shall follow the line of the proof of Fact 1 (see [17], Lemma 2.1) and go a one step further. Write $p_B := \mathbb{P}(X_1 \in B)$ and denote by $(\tau_i, Y_i)_{i \geq 1}$ an i.i.d. sequence for which Y_i is independent of τ_i , $\mathbb{P}(\tau_i = 1) = 1 - \mathbb{P}(\tau_i = 0) = p_B$ and Y_i has the distribution of X_i conditionally to $X_i \in B$. A simple calculation shows that $\mathbb{1}_B(X_i)X_i = \tau_i Y_i$, from where, by conditioning on (τ_1, \dots, τ_n) :

$$\mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow n} \mathbb{1}_B(X_i)X_i\right) \in C\right) = \sum_{\mathcal{P} \subset \{1, \dots, n\}} p_B^{\#\mathcal{P}} (1 - p_B)^{n - \#\mathcal{P}} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow n} \mathbb{1}_{\mathcal{P}}(i)Y_i\right) \in C\right),$$

with the notation $\#\mathcal{P}$ for the number of elements of \mathcal{P} .

Now notice that, for fixed \mathcal{P} and for each permutation of indices σ , the law of $(\mathbb{1}_{\mathcal{P}}(1)Y_1, \dots, \mathbb{1}_{\mathcal{P}}(n)Y_n)$ and $(\mathbb{1}_{\mathcal{P}}(\sigma(1))Y_1, \dots, \mathbb{1}_{\mathcal{P}}(\sigma(n))Y_n)$ are identical. By choosing σ such that $(\mathbb{1}_{\mathcal{P}}(\sigma(1)), \dots, \mathbb{1}_{\mathcal{P}}(\sigma(n))) = (1, \dots, 1, 0, \dots, 0)$ the vector $\sum_{i=1}^{\rightarrow n} \mathbb{1}_{\mathcal{P}}(\sigma(i))Y_i$ has his last $n - \#\mathcal{P} + 1$ coordinates equal, from where, since ϕ is truncating :

$$\mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow n} \mathbb{1}_B(X_i)X_i\right) \in C\right) = \sum_{k=0}^n \binom{n}{k} p_B^k (1 - p_B)^{n-k} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k} Y_i\right) \in C\right).$$

The remainder of the calculus follows exactly as in the proof of Lemma 2.1 in [17], until the last line, where it suffices to prove that

$$\phi\left(\sum_{i=1}^{\rightarrow\eta} \tau_i Y_i\right) =_d \phi\left(\sum_{i=1}^{\rightarrow\eta_B} Y_i\right), \quad (2.1)$$

where η_B is Poisson with expectation np_B , independent of (Y_1, \dots, Y_n) . This is done by writing

$$\begin{aligned} & \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow\eta} \tau_i Y_i\right) \in C\right) \\ &= \sum_{m \geq 0} \mathbb{P}(\eta = m) \sum_{\mathcal{P} \subset \{1, \dots, m\}} p_B^{\#\mathcal{P}} (1 - p_B)^{m - \#\mathcal{P}} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow m} \mathbf{1}_{\mathcal{P}}(i) Y_i\right) \in C\right) \\ &= \sum_{m \geq 0} \frac{n^m}{m!} e^{-n} \sum_{k=0}^m \binom{m}{k} p_B^k (1 - p_B)^{m-k} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k} Y_i\right) \in C\right) \\ & \quad (\text{by the same arguments as above}) \\ &= \sum_{k \geq 0} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k} Y_i\right) \in C\right) \sum_{m \geq k} \frac{n^m}{m!} \frac{m!}{k!(n-k)!} p_B^k (1 - p_B)^{m-k} e^{-n} \\ &= \sum_{k \geq 0} \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k} Y_i\right) \in C\right) e^{-n} \frac{(np_B)^k}{k!} \sum_{m' \geq 0} \frac{(n(1 - p_B))^{m'}}{m'!} \\ &= \sum_{k \geq 0} \mathbb{P}(\eta_B = k) \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k} Y_i\right) \in C\right), \end{aligned}$$

which proves Proposition 2.1. \square

Our next proposition shows that, if ϕ is also zero-irrelevant, then $\phi\left(\sum_{i=1}^{\rightarrow\eta} \mathbf{1}_B(X_i) X_i\right)$, $B \in \mathcal{D}$ has a property of independence which is very similar to the property of independence of Poissonised sums.

Proposition 2.2 *Assume now that ϕ is truncating and zero-irrelevant. In the setting of Proposition 2.1, without imposing that $\mathbb{E}(\eta) = n$, if B_1, B_2, \dots, B_r are disjoint, then*

$$\left[\phi\left(\sum_{i=1}^{\rightarrow\eta} \mathbf{1}_{B_1}(X_i) X_i\right), \dots, \phi\left(\sum_{i=1}^{\rightarrow\eta} \mathbf{1}_{B_r}(X_i) X_i\right) \right] \text{ are mutually independent.}$$

Proof of Proposition 2.2:

Write $\lambda := \mathbb{E}(\eta)$, $p_\ell := \mathbb{P}(X_1 \in B_\ell)$, $\ell = 1, \dots, r$, $p_{r+1} := 1 - \sum_{\ell=1}^r p_\ell$ (assuming without loss of generality that each of these quantities is nonzero) and consider arbitrary events C_1, \dots, C_r . Now define

- For each $i \geq 1$, a mutually independent sequence $(\tau_{i1}, \dots, \tau_{ir})_{i \geq 1}$ for which

- $(\tau_{i1}, \dots, \tau_{ir}) =_d (\mathbb{1}_{B_1}(X_i), \dots, \mathbb{1}_{B_r}(X_i)).$
- A mutually independent family $(Y_{i\ell})_{i \geq 1, \ell=1, \dots, r}$, where the $Y_{i\ell}$ are respectively distributed as $X_i \mid X_i \in B_\ell$.
 - The above-mentioned family are independent from each other.

Direct computations show that, for fixed $i \geq 1$.

$$(\mathbb{1}_{B_1}(X_i)X_i, \dots, \mathbb{1}_{B_r}(X_i)X_i) =_d (\tau_{i1}Y_{i1}, \dots, \tau_{ir}Y_{ir}).$$

We have by conditioning successively with respect to η and $(\tau_{i,\ell})_{i \geq 1, \ell=1, \dots, r}$:

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{\ell=1}^r \left\{ \phi\left(\sum_{i=1}^{\rightarrow \eta} \mathbb{1}_{B_\ell}(X_i)X_i\right) \in C_\ell \right\}\right) \\ &= \mathbb{P}\left(\bigcap_{\ell=1}^r \left\{ \phi\left(\sum_{i=1}^{\rightarrow \eta} \tau_{i\ell}Y_{i\ell}\right) \in C_\ell \right\}\right) \\ &= \sum_{m \geq 0} \mathbb{P}(\eta = m) \mathbb{P}\left(\bigcap_{\ell=1}^r \left\{ \phi\left(\sum_{i=1}^{\rightarrow m} \tau_{i\ell}Y_{i\ell}\right) \in C_\ell \right\}\right) \\ &= \sum_{m \geq 0} \mathbb{P}(\eta = m) \sum_{\substack{\mathcal{P}_1 \cup \dots \cup \mathcal{P}_r \subset \{1, \dots, m\} \\ \mathcal{P}_1, \dots, \mathcal{P}_r \text{ disjoint}}} \prod_{\ell=1}^{r+1} p_\ell^{\#\mathcal{P}_\ell} \mathbb{P}\left(\bigcap_{\ell=1}^r \left\{ \phi\left(\sum_{i=1}^{\rightarrow m} \mathbb{1}_{\mathcal{P}_\ell}(i)Y_{i\ell}\right) \in C_\ell \right\}\right), \end{aligned}$$

with $\mathcal{P}_{r+1} := \{1, \dots, m\} - \bigcup_{\ell=1}^r \mathcal{P}_\ell$ in the preceding formula. Let us focus on a single term of the last sum. Writing $k_\ell := \#\mathcal{P}_\ell$, $\ell = 1, \dots, r+1$, we can find an permutation of indices σ such that the first k_1 integers of $\{1, \dots, m\}$ are $\sigma(i)$, $i \in \mathcal{P}_1$, the next k_2 integers are $\sigma(i)$, $i \in \mathcal{P}_2$, and so on. As ϕ is both truncating and zero-irrelevant, we have almost surely for each $\ell \leq r$ (writing $k_0 := 0$)

$$\phi\left(\sum_{i=1}^{\rightarrow m} \mathbb{1}_{\mathcal{P}_\ell}(\sigma(i))Y_{i\ell}\right) = \phi\left(\sum_{k_0+\dots+k_{\ell-1}}^{\rightarrow k_0+\dots+k_\ell} Y_{i\ell}\right),$$

from where these r random variables are mutually independent. It follows that

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{\ell=1}^r \left\{ \phi\left(\sum_{i=1}^{\rightarrow \eta} \mathbb{1}_{B_\ell}(X_i)X_i\right) \in C_\ell \right\}\right) \\ &= \sum_{m \geq 0} \frac{\lambda^m}{m!} e^{-\lambda} \sum_{\substack{k_1, \dots, k_r \in \{0, \dots, m\} \\ k_{r+1} := m - k_1 - \dots - k_r \geq 0}} m! \prod_{\ell=1}^{r+1} \frac{p_\ell^{k_\ell}}{k_\ell!} \prod_{\ell=1}^r \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k_\ell} Y_{i,\ell}\right) \in C_\ell\right) \\ &= \sum_{k_1, \dots, k_{r+1} \in \mathbb{N}} \prod_{\ell=1}^{r+1} \frac{(\lambda p_\ell)^{k_\ell}}{k_\ell!} e^{-\lambda p_\ell} \prod_{\ell=1}^r \mathbb{P}\left(\phi\left(\sum_{i=1}^{\rightarrow k_\ell} Y_{i,\ell}\right) \in C_\ell\right). \end{aligned}$$

Now writing η_1, \dots, η_r as independent Poisson random variables with respective expectations $\lambda p_1, \dots, \lambda p_r$, which are also independent of $Y_{i,\ell}$, $\ell =$

$1, \dots, r, i \geq 1$, the last expression is equal to

$$\prod_{\ell=1}^r \mathbb{P} \left(\phi \left(\sum_{i=1}^{\rightarrow \eta_\ell} Y_{i\ell} \right) \in C_\ell \right) \sum_{k_{r+1} \geq 0} \frac{(\lambda p_{r+1})^{k_{r+1}}}{k_{r+1}!} e^{-\lambda p_{r+1}} = \prod_{\ell=1}^r \mathbb{P} \left(\phi \left(\sum_{i=1}^{\rightarrow \eta_\ell} Y_{i\ell} \right) \in C_\ell \right).$$

The proof is concluded by applying (2.1) with the formal replacement of η_B, Y_i, τ_i by $\eta_\ell, Y_{i\ell}, \tau_{i\ell}$ for $\ell = 1, \dots, r$. \square

Remarks: First, notice that Fact 1 can be deduced from Proposition 2.1 with the choice of $\phi(d_1, \dots, d_p) := d_p$.

Second, note that, given a collection of functions $\rho_\ell : D \rightarrow \mathbb{R}$ (with $\ell = 1, \dots, r$) the application

$$\phi : (d_1, \dots, d_p) \rightarrow \left[\max_{i=1, \dots, p} \rho_1(d_i), \dots, \max_{i=1, \dots, p} \rho_r(d_i) \right]$$

is truncating. Moreover, if each ρ_ℓ attains its minimum at $0 \in D$, the application ϕ is zero-irrelevant. For particular choices of ρ , we readily obtain two results that may have an interest in themselves.

The choice of $\rho := \pm \mathbf{1}_C$ leads to the following corollary.

Corollary 3.1 *Under the setting of Fact 1 we have :*

$$\mathbb{P} \left(\exists m \leq n, \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \in C \right) \leq 2 \mathbb{P} \left(\exists m \leq \eta, \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \in C \right),$$

$$\mathbb{P} \left(\forall m \leq n, \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \in C \right) \leq 2 \mathbb{P} \left(\forall m \leq \eta, \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \in C \right).$$

Now, dropping the assumption $\mathbb{E}(\eta = n)$ and taking $B_1, \dots, B_r, C_1, \dots, C_r \in \mathcal{D}$ with B_1, \dots, B_r disjoint we have:

(1) If $0 \notin \bigcup_{\ell=1}^r C_\ell$, then

$$\mathbb{P} \left(\bigcap_{\ell=1}^r \left\{ \exists m \leq \eta, \sum_{i=1}^m \mathbf{1}_{B_\ell}(X_i) X_i \in C_\ell \right\} \right) = \prod_{\ell=1}^r \mathbb{P} \left(\exists m \leq \eta, \sum_{i=1}^m \mathbf{1}_{B_\ell}(X_i) X_i \in C_\ell \right),$$

(2) If $0 \in \bigcap_{\ell=1}^r C_\ell$, then

$$\mathbb{P} \left(\bigcap_{\ell=1}^r \left\{ \forall m \leq \eta, \sum_{i=1}^m \mathbf{1}_{B_\ell}(X_i) X_i \in C_\ell \right\} \right) = \prod_{\ell=1}^r \mathbb{P} \left(\forall m \leq \eta, \sum_{i=1}^m \mathbf{1}_{B_\ell}(X_i) X_i \in C_\ell \right).$$

Next, the choice of $\rho(\cdot)$ as a semi norm leads to

Corollary 3.2 *Under the setting of Fact 1 we have, if $(D, \|\cdot\|)$ is a semi*

normed space for which $\|\cdot\|$ is \mathcal{D} measurable :

$$\mathbb{P}\left(\max_{m \leq n} \left\| \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \right\| \in C\right) \leq 2 \mathbb{P}\left(\max_{m \leq \eta} \left\| \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \right\| \in C\right),$$

$$\mathbb{E}\left(H\left(\max_{m \leq n} \left\| \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \right\|\right)\right) \leq 2 \mathbb{E}\left(H\left(\max_{m \leq \eta} \left\| \sum_{i=1}^m \mathbf{1}_B(X_i) X_i \right\|\right)\right).$$

Now dropping the assumption $\mathbb{E}(\eta = n)$ and taking with B_1, \dots, B_r disjoint, the random variables

$$\left[\max_{m \leq \eta} \left\| \sum_{i=1}^m \mathbf{1}_{B_1}(X_i) X_i \right\|, \dots, \max_{m \leq \eta} \left\| \sum_{i=1}^m \mathbf{1}_{B_r}(X_i) X_i \right\| \right]$$

are mutually independent.

Note that the last statement of Corollary 3.2 can be deduced more straightforwardly by making use of the independence properties of Poisson point processes.

Roughly speaking, the preceding corollary shows that blocking arguments for partial sums of i.i.d. random variables can be Poissonised (for which we still have independence properties of Poisson measures). In our proof of Theorem 2 we shall use the particular function ϕ defined as follows. The semigroup D will be taken to be $D([0, 1]^d)^{[0, 1]^d}$, $\chi := [0, \infty)^{[0, 1]^d}$ and

$$\phi(d_1, \dots, d_p) := \left[\max_{i=1, \dots, p} \|d_i(u)\|_{[0, 1]^d} \right]_{u \in [0, 1]^d}.$$

3 Proof of Theorem 1

Choose $\delta > 0$ and a hypercube $I \subset [0, 1 - \delta]^d$ for which $\lambda(I) > 0$. We can assume without loss of generality that $\lambda(I) < 1/2$. By a finite union argument, the full version of Theorem 1 shall readily follow.

Making use of the usual tools in the theory of weak convergence in $D([0, 1]^d)$ (see, e.g., [26], Chapter 1.5) together with standard arguments of countable union/intersection of events, we need to establish the following proposition (in what follows we write $|u|_d$ for $\max\{|u_k|, k = 1, \dots, d\}$).

Proposition 3.1 *For each integer $p \geq 1$, $\theta_1, \dots, \theta_p \in \mathbb{R}^p$ and $t_1, \dots, t_p \in [0, 1]^d$ we have*

$$\int_I \exp\left(i \sum_{j=1}^p \theta_j \Delta_n(u, a_n, t_j)\right) du \rightarrow_{a.s.} \lambda(I) \exp\left(-\frac{1}{2} \theta \Sigma \theta\right), \quad (3.1)$$

where $\Sigma[k, k'] := \text{Cov}(W(t_k), W(t_{k'})) = \lambda([0, t_k] \cap [0, t_{k'}])$.
For each $\epsilon > 0$, there exists $\delta > 0$ such that, almost surely

$$\limsup_{n \rightarrow \infty} \lambda \left(\left\{ u \in I, \sup_{|s-t|_d < \delta} |\Delta_n(u, a_n, s) - \Delta_n(u, a_n, t)| > \epsilon \right\} \right) \leq \epsilon. \quad (3.2)$$

To prove this proposition, we shall apply the following result. For measurability concerns, we shall endow the space $D([0, 1]^d)$ with the σ -algebra \mathcal{T} spawned by the applications :

$$P_{t_1, \dots, t_p}(f) := (f(t_1), \dots, f(t_p)), \quad p \geq 1, \quad t_1, \dots, t_p \in [0, 1]^d.$$

Clearly \mathcal{T} coincides with the σ algebra spawned by the balls related to the norm $\|\cdot\|_{[0,1]^d}$. We shall also consider the Poissonised version of $\Delta_n(\cdot, \cdot)$, namely

$$\Delta \Pi_n(u, a, \cdot) := \frac{\sum_{i=1}^{\eta_n} (\mathbb{1}_{[u, u+a \cdot]}(U_i) - \lambda([0, a \cdot]))}{\sqrt{na^d}}, \quad (3.3)$$

where η_n is a Poisson random variable with expectation n and independent of (U_1, \dots, U_n) . Our proof of Proposition 3.1 relies on the following Proposition.

Proposition 3.2 *Let ρ_n be a sequence of measurable applications from $(D([0, 1]^d), \mathcal{T})$ to \mathbb{C} . Then*

$$\begin{aligned} & \mathbb{E} \left(\left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta_n(u, a_n, \cdot)) du - \mathbb{E} \left(\rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right) \right)^2 \right) \\ &= O(a_n) \text{Var} \left(\rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right). \end{aligned}$$

Proof: We shall apply Fact 1. We choose the semigroup D to be the space $D([0, 1]^d)$, endowed with the σ -algebra $\mathcal{D} := \mathcal{T}$. Clearly the applications of the form

$$\Psi : f \rightarrow \int_I \rho(f([u, u+a \cdot])) du, \quad a \in [0, 1], \quad \rho \text{ measurable from } (D([0, 1]^d), \mathcal{T}) \text{ to } \mathbb{C}.$$

are \mathcal{T} measurable.

We take $B := \{f \in D([0, 1]^d), f(I + [0, a_n]^d) + \lambda(I + [0, a_n]^d) > 0\}$ and $X_i := \mathbb{1}_{[0, \cdot]}(U_i) - \cdot$. Clearly, X_i are all \mathcal{T} measurable and $\mathbb{P}(X_1 \in B) = \mathbb{P}(U_1 \in$

$I + [0, a_n]^d \leq 1/2$ (for all large n). We then consider the applications

$$H_n : f \rightarrow \left(\int_I \frac{1}{\lambda(I)} \rho_n \left(\frac{f([u, u + a_n])}{\sqrt{na_n^d}} \right) du - \mathbb{E} \left(\rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right) \right)^2,$$

which satisfies, for all $n \geq 1$

$$H_n \left(\sum_{i=1}^n \mathbb{1}_B(X_i) X_i \right) =_{a.s.} H_n \left(\sum_{i=1}^n X_i \right).$$

Applying Fact 1 for fixed n leads to the bound

$$\begin{aligned} & \mathbb{E} \left(\left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)) du - \mathbb{E} \left(\rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right) \right)^2 \right) \\ & \leq 2 \mathbb{E} \left(\left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)) du - \mathbb{E} \left(\rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right) \right)^2 \right) \\ & = 2 \text{Var} \left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)) du \right), \text{ as soon as } I + [0, a_n]^d \subset [0, 1]^d, \\ & = 2 \int_I \int_I \text{Cov} \left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)), \frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(v, a_n, \cdot)) \right) dudv \end{aligned}$$

Now, for fixed u, v satisfying $[u, u + a_n] \cap [v, v + a_n] = \emptyset$, the corresponding covariance is null, as $\Delta \Pi_n(u, a_n, \cdot) \perp \Delta \Pi_n(v, a_n, \cdot)$ (this can be seen for example by choosing $B_1 := \{f \in D([0, 1]^d), f([u, u + a_n]) + \lambda([u, u + a_n]) > 0\}$, $B_2 := \{f \in D([0, 1]^d), f([v, v + a_n]) + \lambda([v, v + a_n]) > 0\}$ and $\phi(d_1, \dots, d_p) := d_p$ and applying Proposition 2.2). This entails :

$$\begin{aligned} & \int_I \int_I \text{Cov} \left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)), \frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(v, a_n, \cdot)) \right) dudv \\ & = \int_{\substack{u, v \in I^2 \\ |u-v| \leq a_n}} \text{Cov} \left(\frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)), \frac{1}{\lambda(I)} \int_I \rho_n(\Delta \Pi_n(v, a_n, \cdot)) \right) dudv \\ & \leq \frac{1}{\lambda(I)^2} \int_{\substack{u, v \in I^2 \\ |u-v| \leq a_n}} \sqrt{\text{Var} \left(\int_I \rho_n(\Delta \Pi_n(u, a_n, \cdot)) \right)} \sqrt{\text{Var} \left(\int_I \rho_n(\Delta \Pi_n(v, a_n, \cdot)) \right)} dudv \\ & = O \left(a_n \text{Var} \left(\int_I \rho_n(\Delta \Pi_n(0, a_n, \cdot)) \right) \right). \square \end{aligned}$$

Proof of Proposition 3.1 The following fact shall be needed to prove Proposition 3.1. To the best of our knowledge, it has not yet been written in the literature. However, it can be readily proved by making use of modern tools in empirical processes theory.

Fact 2 *The sequence $\Delta \Pi_n(0, a_n, \cdot)$ converges in distribution to W in the space $(D([0, 1]^d), \|\cdot\|)$.*

The proof of Proposition 3.1 will be achieved in two steps.

Step 1: proof of Proposition 3.1 along a subsequence

Consider the subsequence

$$n_k := \left\lceil \exp(k/\log(k)) \right\rceil, \quad (3.4)$$

so that $\sum_{k \geq 1} a_{n_k} < \infty$ by assumption (1.6). Here $[a]$ stands for the unique integer m fulfilling $m \leq a < m + 1$. By taking, for arbitrary $p \geq 1$, $t_1, \dots, t_p \in [0, 1]^d$ and $\theta_1, \dots, \theta_p \in \mathbb{R}$ the function

$$\rho_n := f \rightarrow \exp\left(i \sum_{j=1}^p \theta_j f(t_j)\right),$$

we readily obtain by Proposition 3.2 that, almost surely

$$\lim_{k \rightarrow \infty} \int_I \exp\left(i \sum_{j=1}^p \theta_j \Delta_{n_k}(u, a_{n_k}, t_j)\right) du - \lambda(I) \mathbb{E}\left(\exp\left(i \sum_{j=1}^p \theta_j \Delta \Pi_{n_k}(0, a_{n_k}, t_j)\right)\right) = 0,$$

from where point (3.1) of Proposition 3.1 is proved along $(n_k)_{k \geq 1}$, making use of Fact 2.

We now fix $\epsilon > 0$ and choose, for fixed $\delta > 0$:

$$\rho_n : f \in D([0, 1]^d) \rightarrow \mathbf{1}_{(\epsilon, \infty)}\left(\sup_{|t' - t|_d < \delta} |f(t') - f(t)|\right).$$

Then, by Proposition 3.2 :

$$\begin{aligned} & \mathbb{E}\left(\left(\frac{1}{\lambda(I)} \lambda\left(\left\{u \in I, \sup_{|t' - t|_d < \delta} |\Delta_n(u, a_n, t') - \Delta_n(u, a_n, t)| > \epsilon\right\}\right)\right.\right. \\ & \quad \left.\left.- \mathbb{P}\left(\sup_{|t' - t|_d < \delta} |\Delta \Pi_n(0, a_n, t') - \Delta \Pi_n(0, a_n, t)|\right)\right)^2\right) = O(a_n). \end{aligned} \quad (3.5)$$

Again, Proposition 2 entails, almost surely :

$$\begin{aligned} & \lim_{k \rightarrow \infty} \lambda\left(\left\{u \in I, \sup_{|t' - t|_d < \delta} |\Delta_{n_k}(u, a_{n_k}, t') - \Delta_{n_k}(u, a_{n_k}, t)| > \epsilon\right\}\right) \\ & = \mathbb{P}\left(\sup_{|t' - t|_d < \delta} |W(t') - W(t)| > \epsilon\right), \end{aligned} \quad (3.6)$$

which proves point (3.2) of Proposition 3.1 along $(n_k)_{k \geq 1}$, as W admits a uniformly continuous version on $[0, 1]^d$.

Step 2: Blocking arguments

Now consider the block $N_k := \{n_{k-1} + 1, \dots, n_k\}$. As $n_k/n_{k-1} \rightarrow 1$ and

$a_{n_k}/a_{n_{k-1}} \rightarrow 1$, we just need to prove (3.1) by replacing $\Delta_n(\cdot, a_n, \cdot)$ by

$$\overline{\Delta}_n(u, s) := \frac{\sum_{i=1}^n \left(\mathbf{1}_{[u, u+a_{n_k}v]}(U_i) - \lambda([0, a_{n_k}s]) \right)}{\sqrt{n_k a_{n_k}^d}}, \quad k \geq 1, \quad n \in N_k, \quad (3.7)$$

which satisfies $\overline{\Delta}_{n_k}(\cdot, \cdot) = \Delta_{n_k}(\cdot, a_{n_k}, \cdot)$ almost surely for each $k \geq 1$. Notice that, for $n \in N_k$, $p \geq 1$, $t_1, \dots, t_p \in [0, 1]^d$, $\theta_1, \dots, \theta_p \in \mathbb{R}$:

$$\begin{aligned} & \left| \int_I \exp \left(i \sum_{j=1}^p \theta_j \overline{\Delta}_n(u, t_j) \right) du - \int_I \exp \left(i \sum_{j=1}^p \theta_j \overline{\Delta}_{n_k}(u, t_j) \right) du \right| \\ & \leq \max_{j=1, \dots, p} |\theta_j| \left(\int_I \left\| \overline{\Delta}_n(u, \cdot) - \overline{\Delta}_{n_k}(u, \cdot) \right\|_{[0,1]^d} du \right). \end{aligned} \quad (3.8)$$

Moreover, for fixed $\epsilon > 0$ and $\delta > 0$ we have almost surely :

$$\begin{aligned} & \lambda \left(\left\{ u : \sup_{|t'-t|_d \leq \delta} \left| \overline{\Delta}_n(u, t') - \overline{\Delta}_n(u, t) \right| > 4\epsilon \right\} \right) \\ & \leq \lambda \left(\left\{ u : \sup_{|t'-t|_d \leq \delta} \left| \overline{\Delta}_{n_k}(u, t') - \overline{\Delta}_{n_k}(u, t) \right| > \epsilon \right\} \right) \\ & \quad + \lambda \left(\left\{ u : \left\| \overline{\Delta}_{n_k}(u, \cdot) - \overline{\Delta}_n(u, \cdot) \right\|_{[0,1]^d} > \epsilon \right\} \right), \end{aligned}$$

where the almost sure limit of the first term is known by (3.6). It turns out that the proof of Proposition 3.1 shall be completed if we can show that

$$\max_{n \in N_k} \int_I \left\| \overline{\Delta}_n(u, \cdot) - \overline{\Delta}_{n_k}(u, \cdot) \right\|_{[0,1]^d} du \xrightarrow{a.s.} 0. \quad (3.9)$$

By making use of the Montgomery-Smith maximal inequality (see [22], Theorem 1 and Corollary 4), we know that, for fixed $\epsilon > 0$:

$$\begin{aligned} & \mathbb{P} \left(\max_{n \in N_k} \int_I \left\| \overline{\Delta}_n(u, \cdot) - \overline{\Delta}_{n_k}(u, \cdot) \right\|_{[0,1]^d} du > 30\epsilon \right) \\ & \leq 9 \mathbb{P} \left(\int_I \left\| \overline{\Delta}_{n_k}(u, \cdot) - \overline{\Delta}_{n_{k-1}}(u, \cdot) \right\|_{[0,1]^d} du > \epsilon \right) \\ & = 9 \mathbb{P} \left(\int_I \left\| \sum_{i=1}^{n_k - n_{k-1}} \mathbf{1}_{[u, u+a_{n_k} \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} du > \epsilon \sqrt{n_k a_{n_k}^d} \right). \end{aligned} \quad (3.10)$$

These probabilities shall be controlled as follows.

Lemma 3 *We have, $\eta_{n_k - n_{k-1}}$ denoting Poisson random variable with expec-*

tation $n_k - n_{k-1}$, independent of $(U_i)_{i \geq 1}$:

$$\mathbb{E} \left(\int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\| du \right) \rightarrow 0, \quad (3.11)$$

$$\begin{aligned} & \mathbb{E} \left(\frac{n_k - n_{k-1}}{n_k} \left(\int_I \left\| \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\| du - \mathbb{E} \left(\int_I \left\| \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\| du \right) \right)^2 \right) \\ &= O(a_{n_k}). \end{aligned} \quad (3.12)$$

Proof: The second point is a straightforward adaptation of Proposition 3.2, while the first point comes from the fact that, for all large k :

$$\begin{aligned} & \mathbb{E} \left(\int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\|_{[0,1]^d} du \right) \\ &= \lambda(I) \times \mathbb{E} \left(\left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta \Pi_{n_k - n_{k-1}}(0, a_{n_k}, \cdot) \right\|_{[0,1]^d} \right) \\ &= \lambda(I) \times \sum_{m \geq 0} \frac{(n_k - n_{k-1})^m}{m!} \exp \left(- (n_k - n_{k-1}) \right) \mu_m, \end{aligned} \quad (3.13)$$

where, $\mu_0 := 0$ and, for each $m \geq 1$:

$$\begin{aligned} \mu_m &:= \mathbb{E} \left(\left\| \sqrt{\frac{m}{n_k}} \Delta_m(0, a_{n_k}, \cdot) \right\|_{[0,1]^d} \right) \\ &\leq C_0 \sqrt{\frac{m}{n_k}}, \end{aligned} \quad (3.14)$$

where C_0 is a universal constant. Note that (3.14) can be proved by a bracketing numbers argument. For example, apply Corollary 19.35, p. 288 in [25] with $\mathcal{F} := \{ \mathbb{1}_{[0, a_n t]}, t \in [0, 1]^d \}$, $F = \mathbb{1}_{[0, a_n]^d}$ and P the uniform distribution on $[0, 1]^d$.

Inserting the bound (3.14) in (3.13) yields

$$\begin{aligned} \mathbb{E} \left(\int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\|_{[0,1]^d} du \right) &\leq \frac{C_0}{\sqrt{n_k}} \mathbb{E} \left(\sqrt{\eta_{n_k - n_{k-1}}} \right) \\ &= O \left(\sqrt{\frac{n_k - n_{k-1}}{n_k}} \right) \\ &= o(1). \square \end{aligned}$$

Now we can prove (3.9) by taking an arbitrary $\epsilon > 0$, applying the bound (3.10), then making use of point (3.11) of Lemma 3 to obtain, for all large k :

$$\begin{aligned}
& \mathbb{P} \left(\int_I \left\| \sum_{i=1}^{n_k - n_{k-1}} \mathbb{1}_{[u, u + a_{n_k} \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\| du > \epsilon \sqrt{n_k a_{n_k}^d} \right) \\
&= \mathbb{P} \left(\int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\| du > \epsilon \right) \\
&\leq \mathbb{P} \left(\left| \int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) \right\| \right. \right. \\
&\quad \left. \left. - \mathbb{E} \left(\int_I \left\| \sqrt{\frac{n_k - n_{k-1}}{n_k}} \Delta \Pi_{n_k - n_{k-1}}(u, a_{n_k}, \cdot) du \right\| \right) \right| > \epsilon/2 \right).
\end{aligned}$$

We then apply point (3.12) of Lemma 3 together with Markov's Inequality and the Borel-Cantelli Lemma. \square

4 Proof of Theorem 2

We first need a large deviation result for $\Delta \Pi_n(0, a_n, \cdot)$. We shall write (recalling (1.4))

$$J(A) := \inf \left\{ J(f), f \in A \right\}, \quad A \subset D([0, 1]^d).$$

Proposition 4.1 *Under the assumptions $a_n \rightarrow 0$ and $na_n / \log \log(n) \rightarrow \infty$ we have :*

- For each closed set $F \in \mathcal{T}$ of $(D([0, 1]^d), \|\cdot\|)$:

$$\limsup_{n \rightarrow \infty} \frac{1}{\log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta \Pi_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \in F \right) \right) \leq -J(F),$$

$$\limsup_{n \rightarrow \infty} \frac{1}{\log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \in F \right) \right) \leq -J(F).$$

- For each open set $O \in \mathcal{T}$ of $(D([0, 1]^d), \|\cdot\|)$:

$$\liminf_{n \rightarrow \infty} \frac{1}{\log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta \Pi_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \in O \right) \right) \geq -J(O),$$

$$\liminf_{n \rightarrow \infty} \frac{1}{\log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \in O \right) \right) \geq -J(O).$$

Proof : The part concerning $\Delta_n(0, a_n, \cdot)$ is a consequence of Proposition 3.2 in [27]. The proof of the part concerning $\Delta \Pi_n(0, a_n, \cdot)$ is very similar to the proof of Proposition 1 in [19]. We omit details. \square

We can assume without loss of generality that $\lambda(I) < 1/2$. The proof shall be split in two parts.

4.1 Upper bounds

This subsection is devoted to proving that, almost surely :

$$\lambda \left(\bigcap_{\epsilon > 0 \in \mathbb{Q}} \bigcap_{n_0 \geq 1} \bigcup_{n \geq n_0} \left\{ u \in I, \frac{\Delta_n(u, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \notin \mathcal{S}^\epsilon \right\} \right) = 0, \quad (4.1)$$

where $\mathcal{S}^\epsilon = \left\{ f \in D([0, 1]^d), \inf \left\{ \|f - g\|_{[0, 1]^d}, g \in \mathcal{S} \right\} < \epsilon \right\}$.

Step 1: proof along a subsequence

Take $(n_k)_{k \geq 1}$ as in (3.4). For fixed $\epsilon > 0$ we shall show that there exists $\delta > 0$ for which, almost surely as $k \rightarrow \infty$:

$$\lambda \left(\left\{ u \in I, \frac{\Delta_{n_k}(u, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}} \notin \mathcal{S}^\epsilon \right\} \right) = O \left(\exp(-(1 + \delta) \log \log(n_k)) \right). \quad (4.2)$$

To achieve this, first notice that, as \mathcal{S} is compact and J is lower semi continuous on $(D([0, 1]^d), \|\cdot\|_{[0, 1]^d})$ we can choose $\delta > 0$ so as $J(D([0, 1]^d) - \mathcal{S}^\epsilon) > 1 + 2\delta$. Moreover, as

$$\liminf_{n \rightarrow \infty} \log(1/a_n) / \log \log(n) > 2,$$

we can assume without loss of generality that $a_n \log(n)^{2+2\delta} \rightarrow 0$. We then make use of Proposition 3.2 with :

$$\phi_n(f) := \exp \left((1 + \delta) \log \log(n) \right) \mathbb{1}_{D([0, 1]^d) - \mathcal{S}^\epsilon} \left(\frac{f}{\sqrt{2na_n^d \log \log(n)}} \right),$$

which yields

$$\begin{aligned} & \mathbb{E} \left(\left(e^{(1+\delta) \log \log(n)} \int_{u \in I} \mathbb{1}_{D([0,1]^d) - \mathcal{S}^\epsilon} \left(\frac{\Delta_n(u, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \right) \right. \right. \\ & \quad \left. \left. - e^{(1+\delta) \log \log(n)} \mathbb{P} \left(\frac{\Delta \Pi_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \notin \mathcal{S}^\epsilon \right) \right)^2 \right) \\ & = O(a_n) \mathbb{P} \left(\frac{\Delta \Pi_n(0, a_n, \cdot)}{\sqrt{2 \log \log(n)}} \notin \mathcal{S}^\epsilon \right) e^{2(1+\delta) \log \log(n)}. \end{aligned}$$

From Proposition 4.1 we know that the last quantity is $o(a_n) \exp((1+\delta) \log \log(n)) = o(\log(n)^{-1-\delta})$ which is sumable along n_k . This proves (4.2) and also proves (4.1) along $(n_k)_{k \geq 1}$.

Step 2 : blocking arguments

Now take $\overline{\Delta}_n(\cdot, \cdot)$ as defined in (3.7). We shall now show that, almost surely

$$\frac{\lambda \left\{ u \in I, \lim_{k \rightarrow \infty} \max_{n \in N_k} \frac{||\overline{\Delta}_n(u, \cdot) - \overline{\Delta}_{n_k}(u, \cdot)||}{\sqrt{2 \log \log(n_k)}} = 0 \right\}}{\lambda(I)} = 1. \quad (4.3)$$

For fixed $k \geq 1$ we shall apply Proposition 2.1 in the following setting: we take semigroup $D := D([0, 1]^d)^{[0, 1]^d}$, endowed with $\mathcal{D} := \mathcal{T} \otimes^{[0, 1]}$. We take $\chi := [0, \infty)^{[0, 1]^d}$ and

$$\phi(d_1, \dots, d_p) := \left[\max_{i=1, \dots, p} ||d_i([u, u + a_{n_k} \cdot])||_{[0, 1]^d} \right]_{u \in [0, 1]^d}.$$

We apply Proposition 2.1 to the sequence $X_m := \mathbb{1}_{[0, \cdot]}(U_m) - \cdot$, $m \geq n_{k-1} + 1$, with $n := n_k - n_{k-1}$ and

$$\begin{aligned} H : g & \rightarrow \left(\frac{\int_{u \in I} \mathbb{1}_{A_k}(g(u)) du}{\lambda(I)} - \mathbf{m}_k \right)^2, \text{ where} \\ A_k & := [\sqrt{2n_k a_{n_k}^d \log \log(n_k)} \epsilon, +\infty), \\ \mathbf{m}_k & := \mathbb{E} \left(\frac{1}{\lambda(I)} \int_I \mathbb{1}_{A_k} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[u, u+a_{n_k} \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0, 1]^d} \right) du \right). \end{aligned}$$

Clearly, if we take the event $B_k := \left\{ f \in D([0, 1]^d), f(I + [0, a_{n_k}]^d) + \lambda(I + [0, a_{n_k}]^d) > 0 \right\}$ we have $\mathbb{P}(X_i \in B) = \mathbb{P}(U_i \in I + [0, a_{n_k}]^d) < 1/2$ (for k large enough), and, for all $\mathbf{n} \geq 1$:

$$H \left(\phi \left(\sum_{n_{k-1}+1}^{\rightarrow n_{k-1}+\mathbf{n}} \mathbb{1}_{B_k}(X_i) X_i \right) \right) =_{a.s.} H \left(\phi \left(\sum_{n_{k-1}+1}^{\rightarrow n_{k-1}+\mathbf{n}} X_i \right) \right),$$

From where, by Proposition 2.1 (writing $v_k := \epsilon \sqrt{2n_k a_{n_k}^d \log \log(n_k)}$)

$$\begin{aligned}
& \mathbb{E} \left(\left(\frac{\lambda \left(\left\{ u \in I, \max_{n \in N_k} \left\| \overline{\Delta}_n(u, \cdot) - \overline{\Delta}_{n_k}(u, \cdot) \right\|_{[0,1]^d} > \epsilon \sqrt{2 \log \log(n_k)} \right\} \right)}{\lambda(I)} - \mathbf{m}_k \right)^2 \right) \\
& \leq \frac{2}{\lambda(I)^2} \text{Var} \left(\int_I \mathbb{1}_{[v_k, +\infty)} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \right) du \right) \\
& = \frac{2}{\lambda(I)^2} \int_{I^2} \text{Cov} \left(\mathbb{1}_{[v_k, +\infty)} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \right), \right. \\
& \quad \left. \mathbb{1}_{[v_k, +\infty)} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[v, v+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \right) \right) dudv.
\end{aligned}$$

For fixed v fulfilling $|s - t| > a_{n_k}$, the corresponding covariance is null, as the two involved random variables are independent. To see this, apply Proposition 2.2 with $B_1 := \{f \in D([0, 1]^d), f([u, u + a_{n_k}]) + \lambda([u, u + a_{n_k}]) > 0\}$ and $B_2 := \{f \in D([0, 1]^d), f([v, v + a_{n_k}]) + \lambda([v, v + a_{n_k}]) > 0\}$. As $\sum_k a_{n_k} < \infty$, assertion (4.3) will be proved as soon as we prove that

$$\sum_{k \geq 1} \mathbb{P} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[0, a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \geq v_k \right) < \infty. \quad (4.4)$$

To show (4.4) we split the probabilities in two

$$\begin{aligned}
\mathbb{P}_k &:= \mathbb{P} \left(\max_{m \leq \eta_{n_k - n_{k-1}}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \geq v_k \right) \\
&\leq \mathbb{P} \left(\max_{m \leq 2(n_k - n_{k-1})} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k} \cdot]) \right\|_{[0,1]^d} \geq v_k \right) \\
&\quad + \mathbb{P} \left(\eta_{n_k - n_{k-1}} > 2(n_k - n_{k-1}) \right).
\end{aligned}$$

The second term is sumable by the Chernoff bound. The first term can be bounded by a sequence of the form

$$s_k := \exp(-A \frac{n_k}{n_k - n_{k-1}} \log \log(n_k)) + \exp(-B \sqrt{n_k a_{n_k}^d \log \log(n_k)}),$$

by making use an inequality of Talagrand (see Inequality A.1 in [16]), with $M = 1$, $\mathcal{G} := \{\mathbb{1}_{[0, a_{n_k} s]}, s \in [0, 1]^d\}$ and $t = \frac{\epsilon}{2} \sqrt{2n_k a_{n_k}^d \log \log(n_k)}$, together

with the following first moment bound for symetrised empirical processes:

$$\begin{aligned}
& \mathbb{E} \left(\left\| \epsilon_i \sum_{i=n_{k-1}+1}^{n_{k-1}+2(n_k-n_{k-1})} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) \right\|_{[0,1]^d} \right) \\
& \leq 2\mathbb{E} \left(\left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+2(n_k-n_{k-1})} \mathbb{1}_{[u, u+a_{n_k}, \cdot]}(U_i) - \lambda([0, a_{n_k}, \cdot]) \right\|_{[0,1]^d} \right) \\
& \leq 2C_0 \sqrt{(n_k - n_{k-1}) a_{n_k}^d \log \log(n_k)} \\
& = o(\sqrt{n_k a_{n_k}^d \log \log(n_k)}),
\end{aligned} \tag{4.5}$$

where (4.5) is no more than the already proved inequality (3.14). As $\log \log(n_k) \sim \log(k)$, $(n_k - n_{k-1})/n_k \rightarrow 0$, $n_k a_{n_k}^d / \log \log(n_k) \rightarrow \infty$, we deduce that $\sum_k s_k < \infty$, which implies that $\sum_k \mathbb{P}_k < \infty$.

4.2 Lower bounds

By compactness of \mathcal{S} , we just need to prove that, for fixed $f \in \mathcal{S}$, $n_0 \in \mathbb{N}$ and $\epsilon > 0$, almost surely:

$$\lambda \left(\bigcap_{n \geq n_0} \left\{ u \in I, \Delta_n(u, a_n, \cdot) \notin f^\epsilon \right\} \right) = 0. \tag{4.6}$$

Step 1: proof for a modified sequence

As $f \in \mathcal{S}$, we have $J(f^\epsilon) = 1 - 2\delta$ for some $\delta > 0$. Consider the subsequence $n_k := \left\lceil \exp(k^{1+\delta}) \right\rceil$ and write

$$\tilde{\Delta}_{n_k}(u, t) := \frac{\sum_{i=n_{k-1}+1}^{n_k} \mathbb{1}_{[u, u+a_{n_k}, t]}(U_i) - \lambda([0, a_{n_k}, t])}{\sqrt{2(n_k - n_{k-1}) a_{n_k}^d \log \log(n_k)}},$$

which defines a sequence of mutually independent processes. As $(n_k - n_{k-1}) a_{n_k}^d \sim n_k a_{n_k}^d$, we can use Proposition (4.1) and obtain, for all large k :

$$\mathbb{P} \left(\tilde{\Delta}_{n_k}(0, \cdot) \in f^\epsilon \right) \geq \exp \left(- (1 - \delta) \log \log(n_k) \right) \geq \frac{1}{k^{1-\delta^2}}, \tag{4.7}$$

from where, when $m \rightarrow \infty$:

$$m^{\delta^2} = O \left(\sum_{k=1}^m \mathbb{P} \left(\tilde{\Delta}_{n_k}(0, \cdot) \in f^\epsilon \right) \right).$$

As the preceding events are mutually independent we obtain for a constant C :

$$\begin{aligned}\mathbb{P}\left(\bigcap_{k=1}^m \left\{ \tilde{\Delta}_{n_k}(0, \cdot) \notin f^\epsilon \right\}\right) &\leq \exp\left(-\mathbb{P}\left(\tilde{\Delta}_{n_k}(0, \cdot) \in f^\epsilon\right)\right) \\ &\leq \exp\left(-Cm^{\delta^2}\right).\end{aligned}$$

Hence by Markov's inequality we have, for fixed $\tau > 0$ and k_0 large enough to fulfill $I + [0, a_{n_{k_0}}]^d \subset [0, 1]^d$, and $m \geq k_0$:

$$\begin{aligned}&\mathbb{P}\left(\lambda\left(\bigcap_{k=k_0}^m \left\{ u \in I, \tilde{\Delta}_{n_k}(u, \cdot) \notin f^\epsilon \right\}\right) > \tau\right) \\ &\mathbb{P}\left(\int_{u \in I} \prod_{k=k_0}^m \mathbb{1}_{D([0,1]^d) - f^\epsilon}(\tilde{\Delta}_{n_k}(u, \cdot)) > \tau\right) \\ &\leq \frac{\lambda(I)}{\tau} \mathbb{P}\left(\bigcap_{k=k_0}^m \left\{ \tilde{\Delta}_{n_k}(0, \cdot) \in f^\epsilon \right\}\right) \\ &\leq \frac{\lambda(I)}{\tau} \exp\left(-Cm^{\delta^2}\right),\end{aligned}$$

which is sumable in m , from where we obtain that, almost surely as $m \rightarrow \infty$:

$$\lambda\left(\bigcap_{k=k_0}^m \left\{ u \in I, \tilde{\Delta}_{n_k}(u, \cdot) \notin f^\epsilon \right\}\right) \rightarrow 0,$$

whence, with probability one :

$$\lambda\left(\bigcup_{k \geq 1} \bigcap_{k \geq k_0} \left\{ u \in I, \tilde{\Delta}_{n_k}(u, \cdot) \notin f^\epsilon \right\}\right) = 0. \quad (4.8)$$

Step 2 : proof for the original sequence

In view of (4.8), and since

$$\begin{aligned}\frac{n_k - n_{k-1}}{n_k} &\rightarrow 1, \\ \frac{\Delta_{n_k}(\cdot, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}} &= \sqrt{\frac{n_{k-1}}{n_k}} \frac{\Delta_{n_{k-1}}(\cdot, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}} + \sqrt{\frac{n_k - n_{k-1}}{n_k}} \tilde{\Delta}_{n_k}(\cdot, \cdot),\end{aligned}$$

we just need to show that, almost surely, as $k_0 \rightarrow \infty$:

$$\lambda\left(\bigcup_{k \geq k_0} \left\{ \sqrt{\frac{n_{k-1}}{n_k}} \left\| \frac{\Delta_{n_{k-1}}(u, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}} \right\|_{[0,1]^d} > \epsilon \right\}\right) \rightarrow 0. \quad (4.9)$$

Again, by Markov's inequality we get, for fixed $\tau > 0$ and k_0 large enough :

$$\begin{aligned} & \mathbb{P}\left(\lambda\left(\bigcup_{k \geq k_0} \left\{\sqrt{\frac{n_{k-1}}{n_k}} \left\|\frac{\Delta_{n_{k-1}}(u, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}}\right\|_{[0,1]^d} > \epsilon\right\}\right) \geq \tau\right) \\ & \leq \frac{\lambda(I)}{\tau} \sum_{k=k_0}^{\infty} \mathbb{P}\left(\sqrt{\frac{n_{k-1}}{n_k}} \left\|\frac{\Delta_{n_{k-1}}(0, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}}\right\|_{[0,1]^d} > \epsilon\right). \end{aligned}$$

Now applying Markov's inequality once again, together with the bound (3.14) we get that

$$\begin{aligned} & \sum_{k=k_0}^{\infty} \mathbb{P}\left(\sqrt{\frac{n_{k-1}}{n_k}} \left\|\frac{\Delta_{n_{k-1}}(0, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}}\right\|_{[0,1]^d} > \epsilon\right) \\ & \leq \frac{1}{\epsilon} \sum_{k=k_0}^{\infty} \mathbb{E}\left(\sqrt{\frac{n_{k-1}}{n_k}} \left\|\frac{\Delta_{n_{k-1}}(0, a_{n_k}, \cdot)}{\sqrt{2 \log \log(n_k)}}\right\|_{[0,1]^d}\right) \\ & = O\left(\sum_{k=k_0}^{\infty} \sqrt{\frac{n_{k-1}}{n_k \log \log(n_k)}}\right), \end{aligned}$$

which is sumable in k_0 . This proves (4.9) and hence completes the proof of Theorem 2.

5 Proof of Theorem 3

The proof is in the same vein as the proof of Theorem 2. Hence, to avoid lengthy redundancies we shall only focus on the single technical difference (even if the real novelty of the present proof relies on the non-written methods that mimic the proof of Theorem 2). First, we shall require the following result, which is included in Proposition 3.2. in [21]. Here $\Delta \Pi F_n(\cdot, a_n, \cdot)$ stands for the poissonised version of $\Delta F_n(\cdot, a_n, \cdot)$, namely:

$$\Delta \Pi F_n(u, a_n, t) := \sum_{i=1}^{\eta_n} \mathbf{1}_{[u, u+a_n t]}(U_i).$$

Fact 4 *Under the assumption $na_n^d \sim c \log \log(n)$ we have*

- *For each closed set $F \in \mathcal{T}$ we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{c \log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta \Pi F_n(0, a_n, \cdot)}{c \log \log(n)} \in F \right) \right) \leq -\mathfrak{I}(F);$$

- For each open set $O \in \mathcal{T}$ we have

$$\liminf_{n \rightarrow \infty} \frac{1}{c \log \log(n)} \log \left(\mathbb{P} \left(\frac{\Delta \Pi F_n(0, a_n, \cdot)}{c \log \log(n)} \in O \right) \right) \geq -\mathfrak{I}(O).$$

The proof of Theorem 3 is achieved following the same steps as in the proof of Theorem 2, replacing $\Delta \Pi_n(\cdot, a_n, \cdot) / \sqrt{2n \log \log(n)}$ by $\Delta \Pi F_n(\cdot, a_n, \cdot) / c \log \log(n)$, and \mathcal{S} by Γ_c . The only point where the methodology changes is when proving an analogue of (4.4), namely $\sum \mathbb{P}'_k < \infty$, where

$$\mathbb{P}'_k := \mathbb{P} \left(\max_{m \leq \eta_{n_k} - n_{k-1}} \left\| \sum_{i=n_{k-1}+1}^{n_{k-1}+m} \mathbf{1}_{[0, a_{n_k} \cdot]}(U_i) \right\| \geq c\epsilon \log \log(n_k) \right).$$

But, in this particular case, as all the summed processes on $[0, 1]^d$ are distribution functions of positive measures, we have

$$\mathbb{P}'_k = \mathbb{P} \left(\sum_{i=1}^{\eta_{n_k} - n_{k-1}} \mathbf{1}_{[0, a_{n_k} \cdot]^d}(U_i) \geq c\epsilon \log \log(n_k) \right),$$

where the involved random variable is Poisson with expectation $(n_k - n_{k-1})a_{n_k}^d = o(\log \log(n_k))$. Hence we avoid making use of Talagrand's inequality (which does not provide a strong enough bound when $na_n^d \sim c \log \log(n)$), and just apply the Chernoff bound for Poisson random variables to establish that $\sum \mathbb{P}'_k < \infty$. \square

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